Fast Mixing of Glauber Dynamics in the SK Model up to the AT Scale

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Abstract

This note improves the (β, h) fast-mixing boundary for Glauber dynamics in the SK model established in [1], from $h \sim \beta^2\sqrt{\log \beta}$ to the AT-scale $h \sim \beta\sqrt{\log \beta}$. The argument was generated entirely by GPT-5.6 Sol, building on the framework and insights of [1]; the human author's contribution was confined to the comparatively mundane task of verifying the proof. The result holds for the p-spin model.

Contents

1 Introduction

1.1 Main abstract result

Let μ be a Gibbs measure on $\{\pm 1\}^n$, and fix a reference configuration $\tau \in \{\pm 1\}^n$. For $x \in \{\pm 1\}^n$, write

$$B_\tau(x) := \{i \in [n] : x_i \neq \tau_i\}$$

for its defect set relative to τ, and for $\delta \in (0,1)$ define the high-overlap core

$$\mathcal{A}_{\tau,\delta} := \left\{ x : \ |B_\tau(x)| \leq \frac{\delta n}{2} \right\}.$$

For every pair $x, y \in \mathcal{A}_{\tau,\delta}$, define disagreement set by $D = D(x,y) := \{i : x_i \neq y_i\}, c = x_{D^c} = y_{D^c}$. Since $x_D = -y_D$, define the associated replica-pair measure on $\{\pm 1\}^D$ by

$$\nu_{D,c}(z) \propto \mu(c,z)\,\mu(c,-z).$$

Theorem 1.1 (Informal main theorem)**.** *Under the following assumptions:*

1. *For all admissible pairs (D,c) (as defined in definition 2.2), we have*

$$\mathrm{gap}_{\mathrm{ct}}(\nu_{D,c}) \geq \kappa,$$

2. *There exist $r \in (0,1)$ such that, for every $x \notin \mathcal{A}_{\tau,\delta}$, at least $\frac{1}{2}|B_\tau(x)|$ defect coordinates $i \in B_\tau(x)$ satisfy*

$$\frac{\mu(x)}{\mu(x^{(i)})} \leq r,$$

   where $x^{(i)}$ is obtained from x by flipping coordinate i (toward the reference state τ_i),

the rate-one-per-site heat-bath Glauber dynamics for μ satisfies

$$\mathrm{Var}_\mu(f) \leq \left[\frac{12M(1+r)}{\delta} + \frac{6M^2}{\kappa} \right] \mathcal{E}_\mu(f,f)$$

for every $f : \{\pm 1\}^n \to \mathbb{R}$, where $\chi := \frac{4r}{\delta} < 1, M := \frac{1}{1-\chi}$. In particular,

$$\mathrm{gap}_{\mathrm{ct}}(\mu) \geq \left[\frac{12M(1+r)}{\delta} + \frac{6M^2}{\kappa} \right]^{-1}.$$

1.2 Organization

Section 2 gives the heat-bath conventions and the reference-state geometry. Section 3 proves the abstract criterion through a replica-core aggregation lemma and a one-sided transport lemma. Section 4 records the probabilistic tools used in the applications. Section 5, **??**, and **??** treat the two-spin SK, pure p-spin, and Hopfield models, respectively.

2 Preliminaries

2.1 Heat-bath Glauber dynamics

Let μ be a full-support probability measure on $\{\pm 1\}^n$. For $i \in [n]$, let K_i denote conditional expectation with respect to all coordinates except i:

$$K_i f(x) := \mathbb{E}_\mu[f(X) \mid X_{[n]\setminus\{i\}} = x_{[n]\setminus\{i\}}].$$

The rate-one-per-site continuous-time heat-bath generator and its Dirichlet form are

$$L_\mu := \sum_{i=1}^n (K_i - I), \qquad \mathcal{E}_\mu(f,f) := -\langle f, L_\mu f\rangle_{L^2(\mu)} = \sum_{i=1}^n \mathbb{E}_\mu[(f - K_i f)^2].$$

The continuous-time spectral gap is

$$\mathrm{gap}_{\mathrm{ct}}(\mu) := \inf_{\mathrm{Var}_\mu(f)>0} \frac{\mathcal{E}_\mu(f,f)}{\mathrm{Var}_\mu(f)}.$$

For $x \in \{\pm 1\}^n$, let $x^{(i)}$ be x with coordinate i flipped and define the heat-bath conductance

$$c_\mu(x, x^{(i)}) := \frac{\mu(x)\mu(x^{(i)})}{\mu(x) + \mu(x^{(i)})}.$$

Lemma 2.1 (Edge representation). *For every $f : \{\pm 1\}^n \to \mathbb{R}$,*

$$\mathcal{E}_\mu(f,f) = \frac{1}{2} \sum_{x \in \{\pm 1\}^n} \sum_{i=1}^n c_\mu(x, x^{(i)})[f(x) - f(x^{(i)})]^2.$$

Equivalently, the right-hand side is the sum over unoriented hypercube edges, with each edge counted once.

Proof. Fix i and an outside configuration $u \in \{\pm 1\}^{[n]\setminus\{i\}}$. Let $x_+ = (u, +1)$ and $x_- = (u, -1)$. The contribution of this two-point fiber to $\mathbb{E}[(f - K_i f)^2]$ is

$$[\mu(x_+) + \mu(x_-)] \frac{\mu(x_+)}{\mu(x_+) + \mu(x_-)} \frac{\mu(x_-)}{\mu(x_+) + \mu(x_-)} [f(x_+) - f(x_-)]^2,$$

which is $c_\mu(x_+, x_-)[f(x_+) - f(x_-)]^2$. Summing over fibers proves the claim. □

The lazy random-scan chain is

$$P_{\mathrm{lazy}} := \frac{1}{2} I + \frac{1}{2n} \sum_{i=1}^n K_i.$$

Since

$$I - P_{\mathrm{lazy}} = -\frac{1}{2n} L_\mu,$$

one has

$$\mathrm{gap}(P_{\mathrm{lazy}}) = \frac{\mathrm{gap}_{\mathrm{ct}}(\mu)}{2n}. \tag{1}$$

2.2 Reference configurations and defect sets

Fix $\tau \in \{\pm 1\}^n$. For $B \subseteq [n]$, define $x^B = x^B(\tau)$ by

$$x_i^B = \begin{cases} -\tau_i, & i \in B, \\ \tau_i, & i \notin B. \end{cases}$$

Every $x \in \{\pm 1\}^n$ has a unique representation $x = x^B$. We write

$$\mathrm{def}_\tau(x) := B, \qquad N_\tau(x) := |B|.$$

Fix $\delta \in (0, 1/4)$ and set

$$k_0 := \left\lfloor \frac{\delta n}{2} \right\rfloor, \qquad \mathcal{A}_{\tau,\delta} := \{x : N_\tau(x) \le k_0\}.$$

If $x, y \in \mathcal{A}_{\tau,\delta}$, their disagreement set

$$D(x, y) := \{i : x_i \ne y_i\}$$

satisfies

$$|D(x, y)| \le N_\tau(x) + N_\tau(y) \le 2k_0 \le \delta n. \tag{2}$$

We simply write $N_-(x), \mathcal{A}_{-,\delta}$ for $\tau = \mathbf{1}$.

2.3 Replica-pair measures

For $D \subseteq [n]$, $c \in \{\pm 1\}^{D^c}$, and $z \in \{\pm 1\}^D$, let $x(c, z) \in \{\pm 1\}^n$ denote the configuration with restriction z on D and c on D^c. Define

$$a_z := \mu(x(c, z)), \qquad w_z := a_z a_{-z}, \qquad Z_{D,c} := \sum_{z \in \{\pm 1\}^D} w_z,$$

and the replica-pair measure

$$\nu_{D,c}(z) := \frac{w_z}{Z_{D,c}}. \tag{3}$$

It is always symmetric: $\nu_{D,c}(z) = \nu_{D,c}(-z)$.

Definition 2.2 (Admissible replica link)**.** *A pair (D, c) is (τ, δ)-admissible if $D \ne \varnothing$, $|D| \le \delta n$, and there exists $z \in \{\pm 1\}^D$ such that*

$$x(c, z),\ x(c, -z) \in \mathcal{A}_{\tau,\delta}.$$

The set of admissible links is denoted by $\mathcal{L}_{\tau,\delta}$.

The admissibility restriction is immaterial for quadratic models, where the pair Hamiltonian loses its dependence on c. It is essential for higher-order models: it restricts the common outside configuration to the high-overlap region and prevents adversarial pinnings far from the reference state.

2.4 Two imported high-temperature criteria

For a symmetric matrix A, let

$$\operatorname{wid}(A) := \lambda_{\max}(A) - \lambda_{\min}(A).$$

Proposition 2.3 (EKZ after diagonal shifting)**.** *Let*

$$\nu(x) \propto \exp\left\{\frac{1}{2}x^\top Ax + b^\top x\right\}$$

be an Ising measure on $\{\pm1\}^m$*. If* $\operatorname{wid}(A) \le \rho < 1$*, then*

$$(1-\rho)\operatorname{Var}_\nu(f) \le \mathcal{E}_\nu(f,f)$$

for every f*.*

Proof. Replace A by $A - \lambda_{\min}(A)I$. This does not change the measure on the hypercube and produces a positive semidefinite interaction with operator norm $\operatorname{wid}(A)$. The claim is the Poincaré theorem of Eldan–Koehler–Zeitouni [2]. □

We also use the following consequence of the smooth-Hamiltonian theorem of Anari–Jain–Koehler–Pham–Vuong [3]. We state it in the continuous-time normalization used here.

Proposition 2.4 (Smooth-Hamiltonian Poincaré criterion)**.** *There exist universal constants* $A_0, B_0 > 0$ *such that the following holds. Let*

$$\nu(x) \propto e^{\Phi(x)}, \qquad x \in \{\pm1\}^m,$$

where Φ *is identified with its multilinear extension. If*

$$\max_{x\in\{\pm1\}^m} \|\nabla^2\Phi(x)\|_{\mathrm{op}} \le A_0,$$

then

$$\operatorname{gap}_{\mathrm{ct}}(\nu) \ge \kappa_0, \qquad \kappa_0 := \frac{1}{1+B_0A_0} > 0.$$

Remark 2.5. *The theorem in [3] is stated for random-scan discrete-time Glauber dynamics, whose gap is at least* $1/[m(1+B_0A_0)]$*. Multiplying by* m *converts it to the rate-one-per-site continuous-time normalization.*

3 The abstract one-sided core-transport theorem

3.1 The two certificates

Assumption 3.1 (Replica-core Poincaré certificate)**.** *There exists* $\kappa > 0$ *such that, for every* $(D,c) \in \mathcal{L}_{\tau,\delta}$*,*

$$\operatorname{gap}_{\mathrm{ct}}(\nu_{D,c}) \ge \kappa.$$

For $B \subseteq [n]$ and $i \in B$, define the deletion odds

$$R_i(B) := \frac{\mu(x^B)}{\mu(x^{B\setminus\{i\}})}.$$

Assumption 3.2 (One-sided deletion certificate)**.** *There exist* $\frac{1}{2} \in (0, 1]$ *and* $r > 0$ *such that, for every* B *with* $|B| > k_0$*, the favorable set*

$$F(B) := \{i \in B : R_i(B) \le r\}$$

satisfies

$$|F(B)| \ge \frac{1}{2}|B|.$$

3.2 Replica-core Dirichlet comparison

Lemma 3.3 (Replica-core pair comparison)**.** *Under assumption 3.1, for every* $f : \{\pm 1\}^n \to \mathbb{R}$*,*

$$\sum_{x,y \in \mathcal{A}_{\tau,\delta}} \mu(x)\mu(y)[f(x) - f(y)]^2 \le \frac{4}{\kappa}\mathcal{E}_\mu(f, f).$$

Proof. Fix $(D, c) \in \mathcal{L}_{\tau,\delta}$ and define $g(z) := f(x(c, z))$. Since $\nu_{D,c}(z) = \nu_{D,c}(-z)$,

$$\begin{aligned}\mathbb{E}_{\nu_{D,c}}[g(Z) - g(-Z)]^2 &\le 2\mathbb{E}[g(Z) - \mathbb{E}g]^2 + 2\mathbb{E}[g(-Z) - \mathbb{E}g]^2 \\ &= 4\operatorname{Var}_{\nu_{D,c}}(g) \le \frac{4}{\kappa}\mathcal{E}_{\nu_{D,c}}(g, g).\end{aligned} \tag{4}$$

We next aggregate the local Dirichlet forms. For $i \in D$, let $z^{(i)}$ denote z with coordinate i flipped, and write

$$a = a_z, \quad a' = a_{z^{(i)}}, \quad b = a_{-z}, \quad b' = a_{-z^{(i)}}.$$

By lemma 2.1,

$$Z_{D,c}\mathcal{E}_{\nu_{D,c}}(g, g) = \frac{1}{2}\sum_{z \in \{\pm 1\}^D}\sum_{i \in D}\frac{aa'bb'}{ab + a'b'}[g(z) - g(z^{(i)})]^2. \tag{5}$$

For positive a, a', b, b',

$$\frac{aa'bb'}{ab + a'b'} \le \frac{aa'}{a + a'}(b + b'). \tag{6}$$

Indeed, after cancelling aa' and cross-multiplying, the right-hand side minus the left-hand side has numerator $ab^2 + a'b'^2 \ge 0$.

Sum (5) over admissible (D, c). Fix an oriented original edge $(x, x^{(i)})$. For each $D \ni i$, the corresponding partner weights are

$$b = \mu(x^{\oplus D}), \qquad b' = \mu(x^{\oplus (D \setminus \{i\})}).$$

Over all subsets $D \ni i$,

$$\sum_{D \ni i}(b + b') = \sum_{y : y_i = -x_i}\mu(y) + \sum_{y : y_i = x_i}\mu(y) = 1. \tag{7}$$

Restricting the sum to $|D| \le \delta n$ and to admissible links can only decrease it. Combining (6), (7), and lemma 2.1 gives

$$\sum_{(D,c) \in \mathcal{L}_{\tau,\delta}} Z_{D,c}\mathcal{E}_{\nu_{D,c}}(g, g) \le \mathcal{E}_\mu(f, f). \tag{8}$$

Every ordered pair $x \neq y$ in the core determines a unique disagreement set $D = D(x, y)$, a unique common outside configuration c, and a unique $z = x_D$ with $y_D = -z$. By (2), the resulting link is admissible. Therefore

$$\begin{aligned}
&\sum_{x,y\in\mathcal{A}_{\tau,\delta}} \mu(x)\mu(y)[f(x) - f(y)]^2 \\
&= \sum_{|D|\le\delta n} \sum_{c} \sum_{z: x(c,z), x(c,-z)\in\mathcal{A}_{\tau,\delta}} w_z[f(x, z) - f(x, -z)]^2 \\
&\le \sum_{|D|\le\delta n} \sum_{c} Z_{D,c}\mathbb{E}_{\nu_{D,c}}[g(Z) - g(-Z)]^2 \\
&\le \frac{4}{\kappa} \sum_{(D,c)\in\mathcal{L}_{\tau,\delta}} Z_{D,c}\mathcal{E}_{\nu_{D,c}}(g, g) \le \frac{4}{\kappa}\mathcal{E}_\mu(f, f),
\end{aligned}$$

where the last two steps use (4) and (8). □

3.3 One-sided deletion transport

Assume assumption 3.2. From any B with $|B| > k_0$, choose $i \in F(B)$ uniformly and move to $B \setminus \{i\}$. Repeat until the path first enters $\{|B| \le k_0\}$. If $|B| \le k_0$ initially, the path is empty. Let R_B be the terminal root and Γ_B the random route.

Define

$$\chi := \frac{4r}{\delta}, \qquad M := \frac{1}{1-\chi},$$

and suppose $\chi < 1$.

Lemma 3.4 (Load and root domination)**.** *Let $X \sim \mu$, and conditional on $X = x^B$ generate the route above. Define the visit load*

$$L(B) := \sum_{C\subseteq[n]} \mu(x^C)\mathbb{P}_C(\Gamma_C \text{ visits } B).$$

Then

$$L(B) \le M\mu(x^B), \qquad |B| > k_0.$$

If q denotes the law of the random root R_X, then

$$q(x^B) \le M\mu(x^B), \qquad |B| \le k_0.$$

Consequently,

$$\mu(\mathcal{A}_{\tau,\delta}^c) \le \chi. \tag{9}$$

Proof. The route strictly decreases cardinality. For B outside the core,

$$L(B) = \mu(x^B) + \sum_{\substack{j\notin B\\ j\in F(B\cup\{j\})}} \frac{L(B\cup\{j\})}{|F(B\cup\{j\})|}. \tag{10}$$

We prove the load estimate by downward induction on $|B|$. Suppose it is known for larger sets. For $C = B \cup \{j\}$, favorability gives

$$\mu(x^C) \le r\mu(x^B),$$

and $|F(C)| \geq \frac{1}{2}(|B|+1)$. Hence

$$\begin{aligned} L(B) &\leq \mu(x^B) + Mr\mu(x^B)\frac{n-|B|}{\frac{1}{2}(|B|+1)} \\ &\leq \mu(x^B) + M\frac{4r}{\delta}\mu(x^B) = (1+M\chi)\mu(x^B) = M\mu(x^B), \end{aligned}$$

because $|B|+1 > \delta n/2$ and $M = 1 + M\chi$.

For the root law, if $|B| < k_0$, no path starting outside the core can first enter the core at B, so $q(x^B) = \mu(x^B)$. If $|B| = k_0$, the same recurrence and estimate apply, yielding $q(x^B) \leq M\mu(x^B)$.

Finally, q is supported on the core and has total mass one. Therefore

$$1 = q(\mathcal{A}_{\tau,\delta}) \leq M\mu(\mathcal{A}_{\tau,\delta}),$$

so $\mu(\mathcal{A}_{\tau,\delta}) \geq 1/M = 1-\chi$. □

Lemma 3.5 (Transport energy)**.** *With X and R_X as above,*

$$\mathbb{E}[f(X) - f(R_X)]^2 \leq \frac{4M(1+r)}{\delta}\mathcal{E}_\mu(f,f).$$

Proof. The flow through a directed deletion edge $B \to B \setminus \{i\}$ is

$$\Phi(B,i) = \frac{L(B)}{|F(B)|} \leq \frac{2M\mu(x^B)}{|B|} \leq \frac{4M}{\delta n}\mu(x^B).$$

The heat-bath conductance of the corresponding unoriented edge satisfies

$$c_\mu(x^B, x^{B\setminus\{i\}}) = \frac{\mu(x^B)}{1+R_i(B)} \geq \frac{\mu(x^B)}{1+r}.$$

Thus

$$\Phi(B,i) \leq \frac{4M(1+r)}{\delta n}c_\mu(x^B, x^{B\setminus\{i\}}). \tag{11}$$

Every route has length at most n. Applying Cauchy–Schwarz along each route and then averaging,

$$\begin{aligned} \mathbb{E}[f(X) - f(R_X)]^2 &\leq n\sum_{(B,i)}\Phi(B,i)[f(x^B) - f(x^{B\setminus\{i\}})]^2 \\ &\leq \frac{4M(1+r)}{\delta}\sum_{(B,i)}c_\mu(x^B, x^{B\setminus\{i\}})[f(x^B) - f(x^{B\setminus\{i\}})]^2 \\ &\leq \frac{4M(1+r)}{\delta}\mathcal{E}_\mu(f,f), \end{aligned}$$

where the last sum is over a subset of the unoriented hypercube edges. □

3.4 Main theorem

Theorem 3.6 (One-sided core-transport criterion)**.** *Let μ be a full-support measure on $\{\pm 1\}^n$, let $\tau \in \{\pm 1\}^n$, and fix $\delta \in (0, 1/4)$. Suppose the replica-core certificate assumption 3.1 holds with constant $\kappa > 0$, and the deletion certificate assumption 3.2 holds with parameters $r > 0$. If*

$$\chi := \frac{4r}{\delta} < 1, \qquad M := (1-\chi)^{-1},$$

then, for every $f : \{\pm 1\}^n \to \mathbb{R}$,

$$\mathrm{Var}_\mu(f) \le \left[\frac{12M(1+r)}{\delta} + \frac{6M^2}{\kappa}\right] \mathcal{E}_\mu(f,f). \tag{12}$$

Consequently,

$$\mathrm{gap}_{\mathrm{ct}}(\mu) \ge \left[\frac{12M(1+r)}{\delta} + \frac{6M^2}{\kappa}\right]^{-1}. \tag{13}$$

Moreover,

$$\mu(\mathcal{A}^c_{\tau,\delta}) \le \chi. \tag{14}$$

Proof. Let X, Y be independent with law μ, and independently generate roots R_X, R_Y. Since

$$2\,\mathrm{Var}_\mu(f) = \mathbb{E}[f(X) - f(Y)]^2,$$

and

$$f(X) - f(Y) = [f(X) - f(R_X)] + [f(R_X) - f(R_Y)] + [f(R_Y) - f(Y)],$$

one has

$$2\,\mathrm{Var}_\mu(f) \le 6\mathbb{E}[f(X) - f(R_X)]^2 + 3\mathbb{E}[f(R_X) - f(R_Y)]^2. \tag{15}$$

By lemma 3.5, the first expectation is at most

$$\frac{4M(1+r)}{\delta}\mathcal{E}_\mu(f,f).$$

By lemma 3.4, the root law q satisfies $q \le M\mu$ pointwise on the core. Therefore, by lemma 3.3,

$$\begin{aligned}
\mathbb{E}[f(R_X) - f(R_Y)]^2 &= \sum_{x,y\in\mathcal{A}_{\tau,\delta}} q(x)q(y)[f(x) - f(y)]^2 \\
&\le M^2 \sum_{x,y\in\mathcal{A}_{\tau,\delta}} \mu(x)\mu(y)[f(x) - f(y)]^2 \\
&\le \frac{4M^2}{\kappa}\mathcal{E}_\mu(f,f).
\end{aligned}$$

Substituting into (15) and dividing by two proves (12). The mass estimate is (9). ∎

Corollary 3.7 (Convenient constant form)**.** *If $\chi \le 1/2$, then $M \le 2$ and*

$$\mathrm{Var}_\mu(f) \le \left[\frac{48}{\delta} + \frac{24}{\kappa}\right] \mathcal{E}_\mu(f,f).$$

In particular, if $r \le \delta/16$, and $\kappa \in (0,1]$, then

$$\mathrm{Var}_\mu(f) \le \left(\frac{48}{\delta} + \frac{24}{\kappa}\right) \mathcal{E}_\mu(f,f), \qquad \mathrm{gap}_{\mathrm{ct}}(\mu) \ge \frac{\delta\kappa}{72}.$$

Proof. Under $\chi \le 1/2$, one has $M \le 2$. Also $\chi < 1$ and $\delta < 1/4$ imply $r < 1$, so $1 + r \le 2$. The first assertion follows from (12). Use the sharper displayed constant and

$$\frac{48}{\delta} + \frac{24}{\kappa} = \frac{48\kappa + 24\delta}{\delta\kappa} \le \frac{72}{\delta\kappa}.$$

∎

3.5 Cold-start mixing corollary

Corollary 3.8 (From the continuous-time gap to worst-start mixing)**.** *Suppose theorem 3.6 gives* $\mathrm{gap}_{\mathrm{ct}}(\mu) \geq \gamma$*. Then the lazy random-scan chain satisfies*

$$\mathrm{gap}(P_{\mathrm{lazy}}) \geq \frac{\gamma}{2n}.$$

For every $\varepsilon \in (0,1)$*,*

$$t_{\mathrm{mix}}^{\mathrm{lazy}}(\varepsilon) \leq \frac{2n}{\gamma}\left[\frac{1}{2}\log\frac{1}{\mu_{\min}} + \log\frac{1}{2\varepsilon}\right].$$

Proof. The gap relation is (1). The mixing estimate is the standard L^2-to-total-variation bound for finite lazy reversible chains. □

4 Probabilistic verification toolbox

4.1 ϵ-Nets

We repeatedly use the following standard fact. If A is an $m \times m$ symmetric matrix and $\mathcal{N}$ is a $1/4$-net of the Euclidean unit sphere with $|\mathcal{N}| \leq 9^m$, then

$$\|A\|_{\mathrm{op}} \leq 2 \max_{v \in \mathcal{N}} |v^\top A v|. \tag{16}$$

For $m \leq \delta n$,

$$\binom{n}{m} 9^m 2^m \leq \exp\left\{C\delta n \log\frac{C}{\delta}\right\}.$$

The same entropy scale controls all configurations with at most δn defects:

$$\sum_{j \leq \delta n} \binom{n}{j} \leq \exp\left\{C\delta n \log\frac{C}{\delta}\right\}. \tag{17}$$

4.2 Hanson–Wright

We use the following form of the Hanson–Wright inequality. If ζ has independent mean-zero, unit-variance, uniformly sub-Gaussian coordinates, then for every symmetric matrix A,

$$\mathbb{P}\left(|\zeta^\top A\zeta - \mathbb{E}\zeta^\top A\zeta| > t\right) \leq 2\exp\left\{-c\min\left(\frac{t^2}{\|A\|_F^2}, \frac{t}{\|A\|_{\mathrm{op}}}\right)\right\}. \tag{18}$$

We apply it only to Rademacher vectors; see [4].

4.3 A counting lemma for half-descent arguments

Lemma 4.1. *Let* $L_\delta := \log(C_0/\delta)$ *with* C_0 *a sufficiently large universal constant. Suppose that for each pair* $T \subseteq B \subseteq [n]$ *with* $|B| = k > \delta n/2$ *and* $|T| = \lceil k/2 \rceil$*, an event* $E_{B,T}$ *satisfies*

$$\mathbb{P}(E_{B,T}) \leq e^{-AkL_k}, \qquad L_k := \log\frac{2en}{k},$$

for a sufficiently large universal A*. Then*

$$\mathbb{P}\left(\bigcup_{k>\delta n/2} \bigcup_{|B|=k} \bigcup_{\substack{T \subseteq B \\ |T|=\lceil k/2 \rceil}} E_{B,T}\right) \leq Ce^{-c\delta n L_\delta}.$$

Proof. The number of pairs (B, T) at fixed k is at most

$$\binom{n}{k}\binom{k}{\lceil k/2\rceil} \le \left(\frac{2en}{k}\right)^k = e^{kL_k}.$$

Choosing A large leaves a bound e^{-2kL_k} at level k. The function $k \mapsto k\log(2en/k)$ is increasing on $[1, n]$, and the geometric sum over $k > \delta n/2$ is at most $Ce^{-c\delta n L_\delta}$. $\square$

5 Application I: the two-spin SK model

5.1 Model and result

Let $W = W^\top$ have independent entries

$$W_{ij} \sim N(0, 1/n), \quad i < j, \qquad W_{ii} = 0.$$

For inverse temperature $\beta \ge 0$ and uniform field h, define

$$\mu_{\beta,h}(x) \propto \exp\left\{\frac{\beta}{2}x^\top W x + h\sum_{i=1}^n x_i\right\}. \tag{19}$$

The reference state is $\tau = \mathbf{1}$.

Theorem 5.1 (Two-spin SK at the AT-order field scale)**.** *There exist universal constants $c, C, c_0 > 0$ such that, for every $\beta \ge 0$, one can choose a deterministic field $h_\beta \ge 0$ satisfying*

$$h_\beta \le C\beta\sqrt{\log(e+\beta)}, \qquad h_\beta = \Theta(\beta\sqrt{\log\beta}) \quad (\beta \to \infty),$$

for which, for every fixed β and all sufficiently large n,

$$\mathbb{P}_W\left(\mathrm{gap}_{\mathrm{ct}}(\mu_{\beta,h_\beta}) \ge \frac{c}{1+\beta^2\log(e+\beta)}\right) \ge 1 - C\exp\left\{-\frac{c_0 n}{1+\beta^2}\right\}.$$

On the same event,

$$\mathrm{gap}(P_{\mathrm{lazy}}) \ge \frac{c}{n[1+\beta^2\log(e+\beta)]},$$

and

$$t_{\mathrm{mix}}^{\mathrm{lazy}}(\varepsilon) \le Cn[1+\beta^2\log(e+\beta)]\left(n[1+\beta\sqrt{\log(e+\beta)}] + \log\frac{1}{\varepsilon}\right).$$

5.2 Replica-core certificate: sparse GOE width

For a quadratic Gibbs measure, the replica-pair identity is exact. If $x(c, z)$ and $x(c, -z)$ differ on D, then the field on D and all interactions between D and D^c cancel:

$$\mu_{\beta,h}(x(c,z))\mu_{\beta,h}(x(c,-z)) = C_{D,c}\exp\{\beta z^\top W_{D,D} z\}. \tag{20}$$

Thus the pair interaction matrix is $2\beta W_{D,D}$.

Let

$$L_\delta := \log\frac{4e}{\delta}.$$

Lemma 5.2 (Uniform sparse GOE norm)**.** *For $\delta \in (0, 1/4)$ and $n \geq 2/\delta$,*

$$\mathbb{P}\left(\max_{|D| \leq \delta n} \|W_{D,D}\|_{\mathrm{op}} > 16\sqrt{\delta L_\delta}\right) \leq 2e^{-14\delta n L_\delta}.$$

Proof. Let $m = \lfloor \delta n \rfloor \geq \delta n/2$. Every smaller principal submatrix can be enlarged to one of size m, so it suffices to consider $|D| = m$. For fixed D and unit $v \in \mathbb{R}^D$,

$$v^\top W_{D,D} v = 2 \sum_{i<j \in D} v_i v_j W_{ij}$$

is centered Gaussian with variance

$$\frac{4}{n} \sum_{i<j} v_i^2 v_j^2 = \frac{2}{n}\left(1 - \sum_i v_i^4\right) \leq \frac{2}{n}.$$

Hence

$$\mathbb{P}(|v^\top W_{D,D} v| \geq t) \leq 2e^{-nt^2/4}.$$

Take a 1/4-net of size at most 9^m, apply (16), and union bound over the $\binom{n}{m}$ sets D. At threshold $16\sqrt{\delta L_\delta}$, the Gaussian exponent dominates the entropy by at least $14\delta n L_\delta$. $\square$

5.3 Deletion certificate: Gaussian half-descent

For a defect set B, write $\sigma^B = x^B(\mathbf{1})$ and define the local field

$$H_i(B) := h + \beta \sum_{j \neq i} W_{ij} \sigma_j^B.$$

For $i \in B$,

$$\frac{\mu_{\beta,h}(\sigma^B)}{\mu_{\beta,h}(\sigma^{B\setminus\{i\}})} = e^{-2H_i(B)}. \tag{21}$$

Lemma 5.3 (Uniform Gaussian half-descent)**.** *If*

$$h \geq 8\beta\sqrt{L_\delta},$$

then

$$\mathbb{P}_W\left(\exists B,\ |B| > \delta n/2 :\ \left|\{i \in B : H_i(B) \geq h/2\}\right| < |B|/2\right) \leq 2e^{-\delta n L_\delta/2}.$$

Proof. Fix B with $|B| = k > \delta n/2$ and put $m = \lceil k/2 \rceil$. If the conclusion fails, there exists $T \subseteq B$, $|T| = m$, such that

$$S_{B,T} := \sum_{i \in T} \sum_{j \neq i} W_{ij} \sigma_j^B < -\frac{hm}{2\beta}.$$

The coefficient of W_{uv} in $S_{B,T}$ is

$$\mathbf{1}_{\{u \in T\}} \sigma_v^B + \mathbf{1}_{\{v \in T\}} \sigma_u^B.$$

Its square is four if both endpoints lie in T, one if exactly one does, and zero otherwise. Therefore

$$\mathrm{Var}(S_{B,T}) = \frac{1}{n}\left(4\binom{m}{2} + m(n-m)\right) \leq 2m.$$

The Gaussian tail gives

$$\mathbb{P}\left(S_{B,T} < -\frac{hm}{2\beta}\right) \leq \exp\left\{-\frac{h^2 m}{16\beta^2}\right\}.$$

The number of pairs (B, T) at level k is at most $(2en/k)^k$. Since $h^2/(16\beta^2) \geq 4L_\delta$ and $\log(2en/k) \leq L_\delta$, a union bound followed by summation over k gives the claim. $\square$

5.4 Parameter choice and proof of theorem 5.1

Fix universal constants

$$\varepsilon_0 := 2^{-20}, \qquad \beta_0 := 2^{-5}.$$

For $\beta \geq \beta_0$, let $\delta_\beta \in (0, 1/4)$ be the unique solution of

$$\delta_\beta \log \frac{4e}{\delta_\beta} = \frac{\varepsilon_0}{\beta^2}, \tag{22}$$

and set

$$h_\beta := 8\beta \sqrt{\log \frac{4e}{\delta_\beta}} + \log \frac{16}{\delta_\beta}. \tag{23}$$

On the event of lemma 5.2, for every $|D| \leq \delta_\beta n$,

$$\operatorname{wid}(2\beta W_{D,D}) \leq 4\beta \|W_{D,D}\|_{\mathrm{op}} \leq 64\beta \sqrt{\delta_\beta L_{\delta_\beta}} = 64\sqrt{\varepsilon_0} = \frac{1}{16}.$$

By proposition 2.3, the replica-core certificate holds with

$$\kappa = 1 - \frac{1}{16} = \frac{15}{16}.$$

By lemma 5.3, at least half of every large defect set has $H_i(B) \geq h_\beta/2$. By (21), these coordinates have odds at most

$$r = e^{-h_\beta} \leq \frac{\delta_\beta}{16}.$$

Thus corollary 3.7 gives

$$\operatorname{gap}_{\mathrm{ct}}(\mu_{\beta,h_\beta}) \geq c\delta_\beta.$$

The failure probability is at most

$$Ce^{-c\delta_\beta n L_{\delta_\beta}} \leq C \exp\left\{-\frac{cn}{\beta^2}\right\}.$$

The parameter equation implies

$$\delta_\beta \asymp \frac{1}{\beta^2 \log(e+\beta)}, \qquad h_\beta \asymp \beta\sqrt{\log(e+\beta)}. \tag{24}$$

Indeed, if $L_\beta = \log(4e/\delta_\beta)$, then

$$L_\beta = \log \frac{4e\beta^2}{\varepsilon_0} + \log L_\beta,$$

so $L_\beta \asymp \log(e+\beta)$.

For $\beta < \beta_0$, set $h_\beta = 0$. A full-matrix net argument gives $\|W\|_{\mathrm{op}} \leq 8$ with probability $1 - e^{-cn}$. Hence

$$\operatorname{wid}(\beta W) \leq 2\beta\|W\|_{\mathrm{op}} < \frac{1}{2},$$

and proposition 2.3 gives $\operatorname{gap}_{\mathrm{ct}} \geq 1/2$.

Finally, on $\|W\|_{\mathrm{op}} \leq 8$,

$$\left|\frac{\beta}{2} x^\top W x + h_\beta \sum_i x_i\right| \leq n(4\beta + h_\beta),$$

so

$$\mu_{\min} \geq \exp\{-n[\log 2 + 8\beta + 2h_\beta]\}.$$

The discrete-time and mixing statements follow from corollary 3.8.

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